Is the Square Root of 2 Irrational?

Preface

Theorem: The square of an even number is even.

Proof: If a is even then $a = 2n$ for some integer n, and therefore

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Theorem: The square of an odd number is odd.

Proof: If a is odd then $a = 2n + 1$ for some integer n, and therefore

$$a^2 = (2n + 1)^2 = 4n^2 + 4n + 1$$

which is **not** evenly divisible by 2. Q.E.D.

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If p were odd, then p^2 would be odd, but from (2) we see that p^2 is even. Therefore p must be even and q must be odd.

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Therefore the assumption that $\sqrt{2}$ is rational must be incorrect. Q.E.D.