Single Most Likely Replacement Detection

John Kormylo

In the echo location problem, one is faced with separating echoes of unknown size and location from background noise. (This differs from the communication problem where one is generally faced with determining which of several possible signals occur at specific times.)

We use a state vector model to produce the source wavelet as its impulse response, and model the reflections as a sparse spike train with Gaussian distributed amplitudes and exponentially distributed intervals between spikes. This corresponds to a signal of the form

$$u(k) = a(k)q(k) \quad \forall \quad k = 0, 1, 2, \ldots, N-1 \tag{1}$$

where the $a(k)$ comprise a zero-mean uncorrelated Gaussian sequence with variance C and the $q(k)$ are a Baysian sequence such that

$$\Pr\{q(k)\} = \begin{cases} \lambda & \text{:q(k) = 1} \\ 1-\lambda & \text{:q(k) = 0} \end{cases} \quad \forall \quad k = 0, 1, \ldots, N-1. \tag{2}$$

The state vector model is given by

$$\mathbf{x}(k+1) = \Phi \mathbf{x}(k) + \boldsymbol{\gamma} u(k) \tag{3}$$

and

$$z(k) = \mathbf{h}' \mathbf{x}(k) + n(k) \tag{4}$$

where $z(k)$ is our observed signal and $n(k)$ is the observation noise modeled as a zero-mean uncorrelated Gausian sequence with variance R.

For a given sequence $q(k)$ for $k = 0, 1, \ldots, N-1$, the variance of the input signal is given by

$$\mathcal{E}\left\{u^2(k)|q(k)\right\} = Cq(k) \tag{5}$$

which can be used by a Kalman smoother to estimate spike amplitudes. Needless to say, this produces far more accurate results than using the average variance

$$\mathcal{E}\left\{u^2(k)\right\} = C\lambda = Q \quad . \tag{6}$$

The only problem is that we don't know the correct $q(k)$ sequence.

For a given sequence $\mathbf{q}' = [q(0), q(1), \ldots, q(N-1)]$, the negative log likelihood function is given by

$$J(\mathbf{q}) = 2m \log\left(\frac{1-\lambda}{\lambda}\right) + \sum_{k=1}^{N} \frac{\tilde{z}^2(k|k-1)}{\mathbf{h}' P(k|k-1)\mathbf{h} + R} + \log\left(\mathbf{h}' P(k|k-1)\mathbf{h} + R\right) \tag{7}$$

where m is the number of spikes. As will be shown in the appendix, the log likelihood ratio for replacing a single $q(k)$ with $1 - q(k)$ is given by

$$\begin{aligned} \log \Lambda(k) = {} & \frac{\left(\boldsymbol{\gamma}' \mathbf{r}(k+1)\right)^2}{C^{-1}(1-2q(k)) + \boldsymbol{\gamma}' S(k+1)\boldsymbol{\gamma}} - \log\left[1 + C(1-2q(k))\boldsymbol{\gamma}' S(k+1)\boldsymbol{\gamma}\right] \\ & - 2(1-2q(k)) \log \frac{\lambda}{1-\lambda} \end{aligned} \tag{8}$$

where $\mathbf{r}(k+1)$ and $S(k+1)$ are produced by the Kalman smoother.

This means that by running a Kalman smoother for a given sequence $\mathbf{q}$ one can obtain the log likelihood ratios for replacing $q(k)$ with $1 - q(k)$ for every possible value of k. The SMLR detector performs the replacement which causes the greatest improvement in likelihood, then repeats the procedure until no beneficial replacements are left.

Appendix

Define vectors $\mathbf{u}' = [u(0), u(1), \ldots, u(N-1)]$ and $\mathbf{z}' = [z(1), z(2), \ldots, z(N)]$, and matrix

$$\Omega = \mathcal{E}\{\mathbf{z}\mathbf{z}'|\mathbf{q}\} \quad . \qquad (A-1)$$

The optimal smoother for a given $\mathbf{q}$ is can be written as

$$\hat{\mathbf{u}}(\mathbf{q}) = \mathcal{E}\{\mathbf{u}\mathbf{z}'|\mathbf{q}\}\,\Omega^{-1}\mathbf{z} \qquad (A-2)$$

and the negative log-likelihood function as

$$J(\mathbf{q}) = z'\Omega^{-1}z + \log\det\Omega - 2m\log\left(\frac{\lambda}{1-\lambda}\right) \qquad (A-3)$$

where $\det\Omega$ denotes the determinant of matrix Ω.

Define triangular matrix

$$V = \begin{pmatrix} v(1) & 0 & 0 & \ldots & 0 \\ v(2) & v(1) & 0 & \ldots & 0 \\ v(3) & v(2) & v(1) & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ v(N) & v(N-1) & v(N-2) & \ldots & v(1) \end{pmatrix} \qquad (A-4)$$

where

$$v(k) = \mathbf{h}'\Phi^{k-1}\boldsymbol{\gamma} \quad \forall \quad k > 0 \qquad (A-5)$$

is the source wavelet. One can now write

$$\mathbf{z} = V\mathbf{u} + \mathbf{n} \qquad (A-6)$$

where $\mathbf{n}' = [n(1), n(2), \ldots, n(N)]$, and therefore

$$\mathcal{E}\{\mathbf{u}\mathbf{z}'|\mathbf{q}\} = \mathcal{E}\{\mathbf{u}\mathbf{u}'|\mathbf{q}\}\,V' \qquad (A-7)$$

and

$$\Omega = V\mathcal{E}\{\mathbf{u}\mathbf{u}'|\mathbf{q}\}\,V' + RI \qquad (A-8)$$

where

$$\mathcal{E}\{\mathbf{u}\mathbf{u}'|\mathbf{q}\} = \begin{pmatrix} Cq(0) & 0 & \ldots & 0 \\ 0 & Cq(1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & Cq(N-1) \end{pmatrix} \quad . \qquad (A-9)$$

Consider sequence $\mathbf{q}_k$ which differs from $\mathbf{q}$ only by replacing $q(k)$ with $1-q(k)$. The corresponding covariance matrix is given by

$$\Omega_k = \mathcal{E}\{\mathbf{z}\mathbf{z}'|\mathbf{q}_k\} = \Omega + (1-2q(k))\mathbf{v}_k C\mathbf{v}_k' \qquad (A-10)$$

where vector $\mathbf{v}_k$ is the kth column of matrix V. The log likelihood ratio for choosing $\mathbf{q}_k$ over $\mathbf{q}$ is given by

$$\begin{aligned}\log \Lambda(k) &= J(\mathbf{q}) - J(\mathbf{q}_k) \\ &= \mathbf{z}'\Omega^{-1}\mathbf{z} - \mathbf{z}'\Omega_k^{-1}\mathbf{z} + \log\left(\frac{\det \Omega}{\det \Omega_k}\right) - 2(1-2q(k))\log\frac{\lambda}{1-\lambda} \quad .\end{aligned} \qquad (A-11)$$

From the matrix inversion lemma we can write

$$\Omega_k^{-1} = \Omega^{-1} - \frac{\Omega^{-1}\mathbf{v}_k\mathbf{v}_k'\Omega^{-1}}{C^{-1}(1-2q(k)) + \mathbf{v}_k'\Omega^{-1}\mathbf{v}_k} \qquad (A-12)$$

and therefore

$$\mathbf{z}'\Omega_k^{-1}\mathbf{z} = \mathbf{z}'\Omega^{-1}\mathbf{z} - \frac{(\mathbf{v}_k\Omega^{-1}\mathbf{z})^2}{C^{-1}(1-2q(k)) + \mathbf{v}_k'\Omega^{-1}\mathbf{v}_k} \quad . \qquad (A-13)$$

One can also show that

$$\begin{aligned}\frac{\det \Omega_k}{\det \Omega} &= \det[\Omega^{-1}\Omega_k] \\ &= 1 + C(1-2q(k))\mathbf{v}_k'\Omega^{-1}\mathbf{v}_k\end{aligned} \qquad (A-14)$$

and the log-likelihood ratio in (A-11) can be simplified to

$$\begin{aligned}\log \Lambda(k) = &\frac{(\mathbf{v}_k\Omega^{-1}\mathbf{z})^2}{C^{-1}(1-2q(k)) + \mathbf{v}_k'\Omega^{-1}\mathbf{v}_k} - \log\left[1 + C(1-2q(k))\mathbf{v}_k\Omega^{-1}\mathbf{v}_k\right] \\ &- 2(1-2q(k))\log\frac{\lambda}{1-\lambda}\end{aligned} \quad . \qquad (A-15)$$

From the optimal smoother formulation in (A-2) using (A-9) we can write

$$\hat{u}(k|\mathbf{q}) = Cq(k)\mathbf{v}_k'\Omega^{-1}\mathbf{z} \qquad (A-16)$$

while the Kalman smoother formulation is given by

$$\hat{u}(k|\mathbf{q}) = Cq(k)\boldsymbol{\gamma}'\mathbf{r}(k+1) \qquad (A-17)$$

which means that

$$\mathbf{v}_k'\Omega^{-1}\mathbf{z} = \boldsymbol{\gamma}'\mathbf{r}(k+1) \quad . \qquad (A-18)$$

The error covariance for (A-2) is given by

$$\mathcal{E}\{\tilde{\mathbf{u}}\tilde{\mathbf{u}}'|\mathbf{q}\} = \mathcal{E}\{\mathbf{u}\mathbf{u}'|\mathbf{q}\} - \mathcal{E}\{\mathbf{u}\mathbf{u}'|\mathbf{q}\}V'\Omega^{-1}V\mathcal{E}\{\mathbf{u}\mathbf{u}'|\mathbf{q}\} \qquad (A-19)$$

and therefore using (A-9) we get

$$\mathcal{E}\left\{\tilde{u}^2(k|\mathbf{q})\right\} = Cq(k) - C^2q(k)\mathbf{v}_k'\Omega^{-1}\mathbf{v}_k \quad . \qquad (A-20)$$

Comparing that with the Kalman smoother formulaton

$$\mathcal{E}\left\{\tilde{u}^2(k|\mathbf{q})\right\} = Cq(k) - C^2 q(k)\boldsymbol{\gamma}' S(k)\boldsymbol{\gamma} \qquad (A-21)$$

we see that

$$\mathbf{v}_k' \Omega^{-1}\mathbf{v}_k = \boldsymbol{\gamma}' S(k)\boldsymbol{\gamma} \quad . \qquad (A-22)$$

Substituting (A-18) and (A-22) back into (A-15) we obtain our desired result

$$\begin{aligned}\log\Lambda(k) = & \frac{\left(\boldsymbol{\gamma}'\mathbf{r}(k+1)\right)^2}{C^{-1}(1-2q(k)) + \boldsymbol{\gamma}' S(k+1)\boldsymbol{\gamma}} - \log\left[1 + C(1-2q(k))\boldsymbol{\gamma}' S(k+1)\boldsymbol{\gamma}\right] \\ & - 2(1-2q(k))\log\frac{\lambda}{1-\lambda}\end{aligned} \quad . \qquad (A-23)$$