Nahi's Balanced Realization

A Tutorial

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System identification refers to obtaining a state vector model for a system from input and output measurements. It is generally a rather nasty problem in nonlinear optimization. Nahi's balanced realization uses a singular value decomposition (which is a much nicer problem in nonlinear optimization) to produce a model which, in my experience, cannot be improved upon.

The first step is to obtain an estimate of the impulse response from input/output data. Design a prediction error filter over the input data and apply it to both the input and output signals. The cross-correlation between the filtered input and output signals will then equal the impulse response, so long as the impulse response is shorter than filter length.

Given the impulse response, $h(k)$ for $k = 0, 1, 2, \ldots, N$, construct a symmetrical matrix of the form

$$H = \begin{bmatrix} h(0) & h(1) & h(2) & \ldots & h(N/2) \\ h(1) & h(2) & h(3) & \ldots & h(N/2+1) \\ h(2) & h(3) & h(4) & \ldots & h(N/2+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(N/2) & h(N/2+1) & h(N/2+2) & \ldots & h(N) \end{bmatrix} \quad . \tag{1}$$

An eigenvalue decomposition of H yields

$$H = V \Lambda V' = \sum_{i=1}^{N/2} \mathbf{v}_i \lambda_i \mathbf{v}_i{}' \tag{2}$$

where Λ is a diagonal matrix and $VV' = V'V = I$. Since H is symmetrical but not positive definite, the eigenvalues will be real but not always positive. If one truncates Λ to the largest (by absolute value) n terms, the modeling error equals the sum of the unused eigenvalues squared, or

$$\sum_{i=1}^{N/2} \sum_{j=1}^{N/2} \left(H_{ij} - \widehat{H}_{ij} \right)^2 = \mathrm{tr}\{(H - \widehat{H})^2\} = \sum_{i=n+1}^{N/2} \lambda_i^2$$

where

$$\widehat{H} = \sum_{i=1}^{n} \mathbf{v}_i \lambda_i \mathbf{v}_i{}' \quad .$$

For a state vector model, the impulse response satisfies

$$h(k) = \mathbf{h}' \Phi^k \gamma \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{3}$$

where $\mathbf{h}$ is the observation vector, Φ is the transition matrix and γ is the input vector. Consequently, matrix H can be written as

$$H = \begin{bmatrix} \mathbf{h}' \\ \mathbf{h}'\Phi \\ \mathbf{h}'\Phi^2 \\ \vdots \\ \mathbf{h}'\Phi^{N/2} \end{bmatrix} \begin{bmatrix} \gamma & \Phi\gamma & \Phi^2\gamma & \ldots & \Phi^{N/2}\gamma \end{bmatrix} \tag{4}$$

which has the same form (more or less) as the truncated eigenvalue decomposition $\widehat{H} = V \hat{\Lambda} V'$ when the number of eigenvalues equals the model order.

The balanced realization assumes

$$VSD = \begin{bmatrix} \mathbf{h}' \\ \mathbf{h}'\Phi \\ \mathbf{h}'\Phi^2 \\ \vdots \\ \mathbf{h}'\Phi^{N/2} \end{bmatrix} \tag{5}$$

and

$$DV' = \begin{bmatrix} \gamma & \Phi\gamma & \Phi^2\gamma & \dots & \Phi^{N/2}\gamma \end{bmatrix} \tag{6}$$

where S and D are real diagonal matrices such that

$$\widehat{\Lambda} = SD^2 \quad \text{and} \quad S^2 = I \quad .$$

In other words, S contains the signs of the eigenvalues and $D = (S\widehat{\Lambda})^{\frac{1}{2}}$.

One can get vectors γ and $\mathbf{h}'$ from the first column of DV' and the first row of VSD, respectively. (As demonstrated in the appendix, this is actually a least squares solution.) Since shifting DV' left one column equals $\Phi DV'$ (except for the last column) we can write

$$DV'T' = \Phi D \left(V' - \begin{bmatrix} 0 & \dots & 0 & \mathbf{r} \end{bmatrix}\right)$$

where shift matrix T is given by

$$T = \begin{bmatrix} \mathbf{0} & I \\ 0 & \mathbf{0}' \end{bmatrix}$$

and $\mathbf{r}$ is the last column of V. Post-multiplying both sides by V yields

$$DV'T'V = \Phi D(I - \mathbf{r}\mathbf{r}')$$

which has the solution

$$\Phi = DV'T'V \left(I + \frac{\mathbf{r}\mathbf{r}'}{1 - \mathbf{r}'\mathbf{r}}\right) D^{-1} \tag{7}$$

using the matrix inversion lemma on $(I - \mathbf{r}\mathbf{r}')^{-1}$.

Appendix

We want to estimate $\mathbf{h}$ to minimize the square error

$$\begin{aligned} J &= \sum_{k=0}^{N/2} \left(h(k) - \mathbf{h}'\Phi^k\gamma\right)^2 \\ &= \left(\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} H - \mathbf{h}'DV'\right)\left(\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} H - \mathbf{h}'DV'\right)' \quad . \end{aligned}$$

Setting the partial derivative with respect to $\mathbf{h}$ to zero gives us

$$DV' \left(H \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - VD\mathbf{h} \right) = \mathbf{0}$$

and therefore

$$DV'H \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = DV'VD\mathbf{h}$$

or

$$D\hat{\Lambda}V' \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = D^2\mathbf{h}$$

since the unused eigenvectors of H are orthogonal to V. The solution is then given by

$$\mathbf{h} = SDV' \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{or} \quad \mathbf{h}' = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} VSD$$

since $\hat{\Lambda} = SD^2$.