# A Fractal Manifold

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The idea is to intermix tubes (holes in a metal matrix) containing two separate fluids as closely as possible. Applications include heat exchangers and burners. The advent of 3D printing technology is what makes these designs practical.

The border of a fractal is infinite in length, although the practical limit is determined by the resolution of the 3D printer. For a heat exchanger, each layer increases the surface area by a factor of $\sqrt{5}$ (Cartesian grid) or by $\sqrt{3}$ (hexagonal grid). For a burner, the smaller the grid spacing the faster the two fluids will mix.

The last rocket engine using a grid burner was the F1 engine, which was subject to a rather extreme problem with combustion instability. The solution consisted of dividing the pressure plate into an inner disk and and outer ring, and rotating the two relative to each other. This would indicate that the combustion instability is directional along the grid lines.

For comparison, the shuttle engines used two "shower heads" aimed at each other, and the Merlin engine combines thin layers of RP1 and LOX moving radially outward.

According to Barlow's formula for the maximum pressure for cylindrical pipes,

$$P = \frac{2\sigma_i s}{D_m}$$

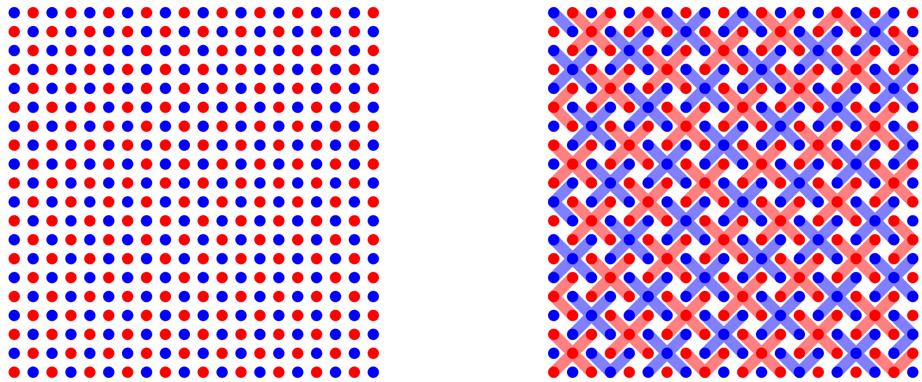
where $P$ is pressure, $\sigma_i$ is allowable stress, $s$ is the wall thickness and $D_m$ is the outside diameter. So long as the wall thickness and tube diameter remain proportional, the strength of the tubes in different layers will remain the same.

## Cartesian Grid

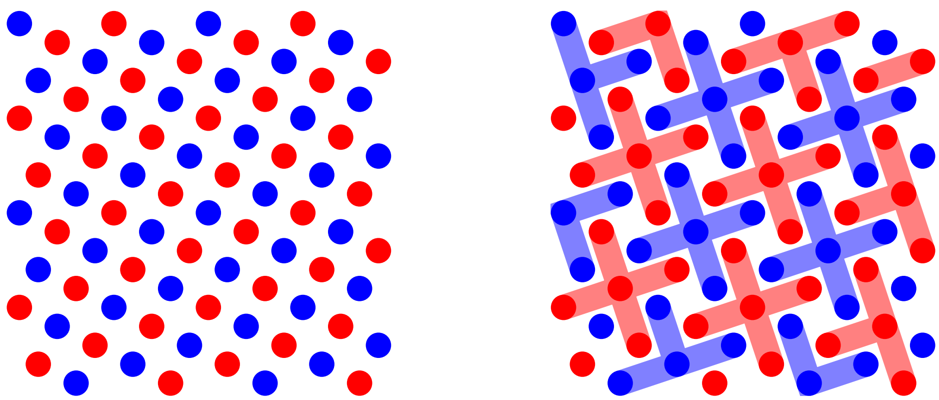
Consider a regular grid of tubes containing two fluids, marked as red and blue. Each red tube is surrounded by four blue tubes, and each blue tube is surrounded by four red tubes, except at the edges.

In the next layer, five of these tubes are connected together in a regular pattern (2 right to 1 down or 2 up to 1 right to change colors, 3 right to 1 up or 3 up to 1 left for same color). This and its mirror image are the only such patterns possible. Note that every dot is connected, excepting some of those at the edge of the grid.

The centers of the tubes are separated by 0.707 ($\sqrt{2}/2$) times the grid spacing, $s$. In this diagram, we used a tube radius of $0.3s$, leaving about $0.107s$ separation between the tube edges.

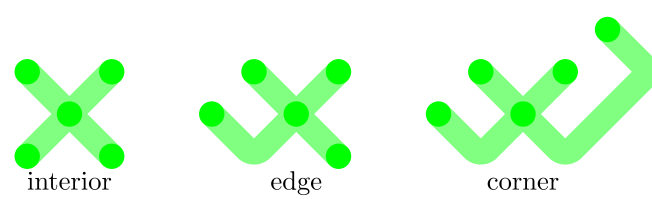


The centers of the connected tubes also form a regular grid, although rotated by about $-26.6^\circ$ ($\tan^{-1} -1/2$). Also, since five tubes are feeding into one, we assume the diameter of the tubes will increase by $\sqrt{5}$, as does the grid spacing. Again, one can connect the tubes together using the same (but rotated) pattern.

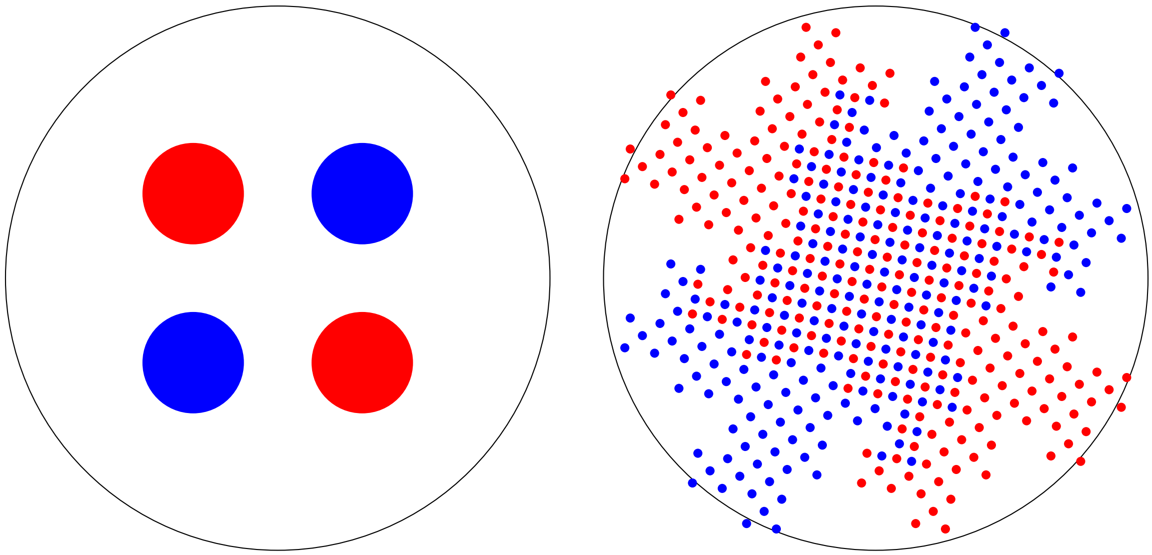


However, something needs to be done at the edges. If you start with a simple grid of 4 tubes and go three layers deep (large to small), the resulting pattern will have large areas with nothing but tubes of one color, not to mention the lacy edges (typical of fractals). Even the useful area forms an irregular (fractal) shape.

The good news is that at the edges one can extend the arms of the connecting tube shapes without running into other tubes. The following shapes have proven effective in compacting the usable area. More exotic shapes are based on these.




Is is not always clear which shape to use. Small changes have big consequences in later

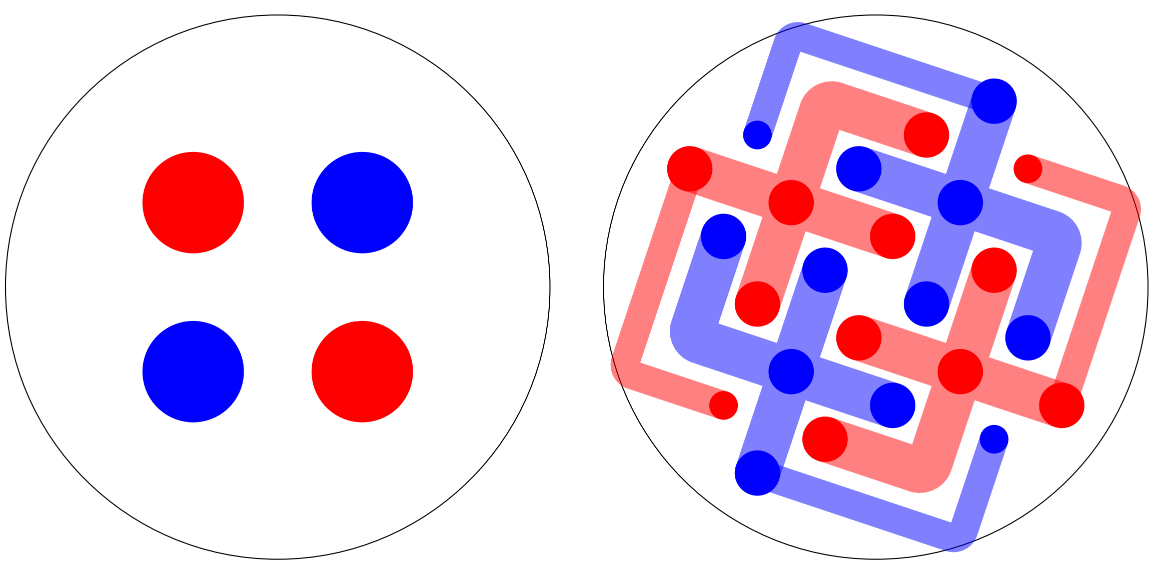


layers.

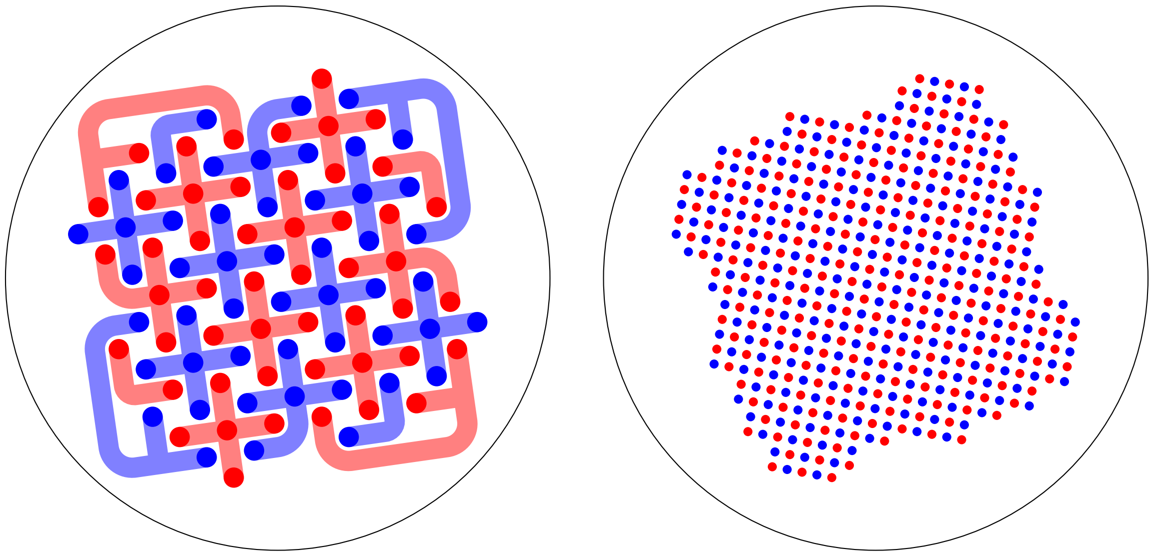
The first rule is to never connect to the next layer if the tube isn't adjacent to at least 2 tubes of the other color. Keep in mind that changing one shape can add or remove good locations for another. The second rule is to never waste a tube, even if you have to extend it quite a distance to find a good location.

You need to move tubes away from "peaks" and toward "valleys." If you have more that 2 corners (tubes adjacent to only 2 tubes of the other color) in a row, you need to break them up to avoid serious problems in the next layer.

Starting from the same $2 \times 2$ grid, four layers are shown below.



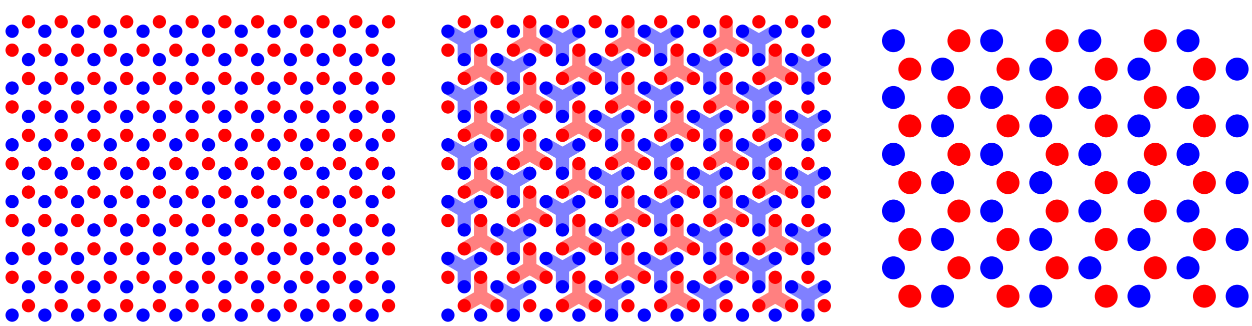
Of particular interest are the corner shapes in the second layer. One arm is connected to



the next layer in two locations. However, only five of the ten possible child tubes are used, so the net flow is the same as a normal arm. Without both dots there would not be any two dot locations available for the other color, and vice verses.

## Hexagonal Grid

In this grid each blue dot is surrounded by 3 red dots and vice verses. While combustion is aligned along 3 directions (30°, 90° and 150°), there is a gap between the combustion zones in any direction, which should reduce the problem with combustion instability.

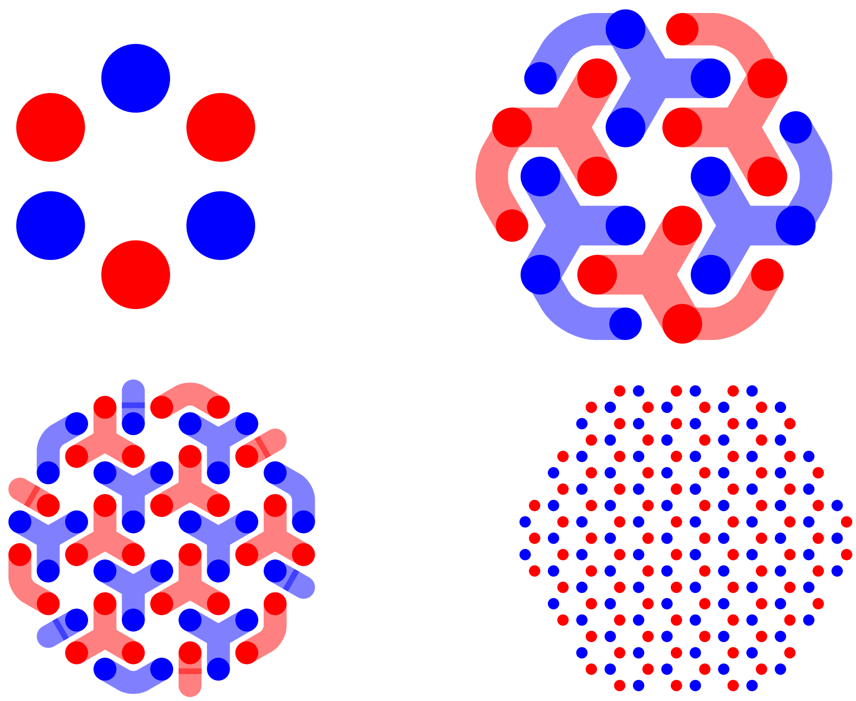


In the next layer three of these tubes are connected together. Unlike the Cartesian grid, the centers are not located on top of a dot. The center-to-center distance between tubes is about $0.866s$ ($\sqrt{3}/2$) where $s$ is the spacing between dots of different colors. This diagram used a radius of $0.35s$ leaving about a $0.166s$ gap between tubes. The centers form another hexagonal grid rotated by $\pm 30°$. The radius and grid spacing both increase by $\sqrt{3}$, so self-symmetry is maintained.

As before, another shape is useful at the edges to keep the grid more compact. Not only does this shape extend an arm, this arm connects to two different tubes in the next layer.

However, in the next layer only three out of six possible tubes are produced from those two connecting tubes, so the net flow is exactly the same as a normal arm.

Starting from a hexagon, the first four layers are shown below. Note that all of them remain roughly hexagonal.



It appears that the special shapes are only used in every other layer. While they have to be near an edge to work, not all tubes near an edge use them. The critical criterion is that they have to be near a nearly completed hexagon.

## Junctions between Layers

I suspect there are CAD-CAM packages which do this sort of thing, but not having access one I decided to do the math myself. In particular, there isn't enough room to fit a $0.3\sqrt{5}s$ or $0.35\sqrt{3}s$ radius tube all the way through where the five or three tubes join. What we need is the smallest tube which can handle the flow.

The net flow from a horizontal cylinder added at any depth $z$ is proportional to the cumulative area of a circle from bottom to top, which is given by

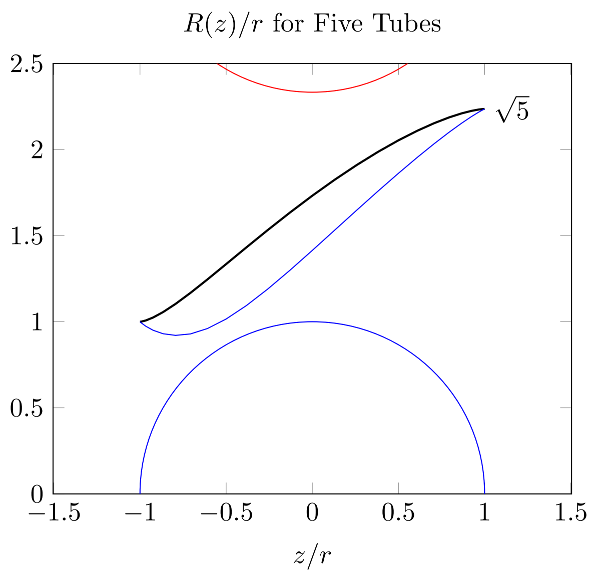
$$\int_{-r}^{z} 2\sqrt{r^2 - x^2}\,dx = r^2 \int_{-\pi/2}^{\sin^{-1}(z/r)} 2\cos^2\theta\,d\theta = r^2 \left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\sin^{-1}(z/r)}$$

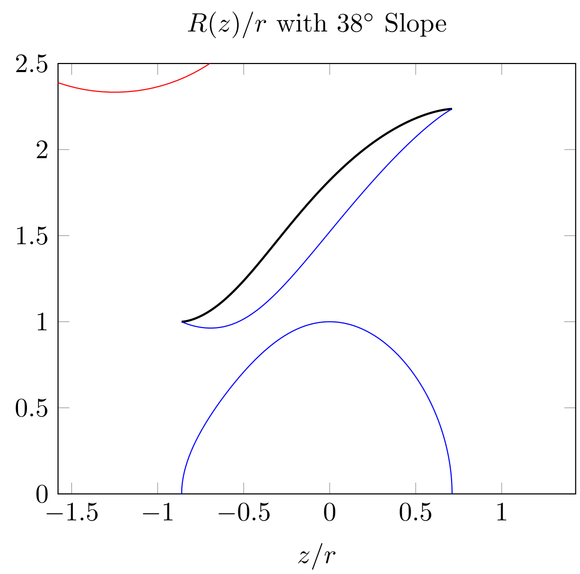
which is easier to handle parametrically with $z = r\sin\theta$. At the junction of five tubes (Cartesian grid) the minimum radius $R(z)$ needed to handle the flow is given by:

$$R(z) = r\sqrt{1 + 4\left(\frac{\theta}{\pi} + \frac{1}{2} + \frac{\sin 2\theta}{2\pi}\right)}\ \Bigg|_{\theta=\sin^{-1}(z/r)}$$

where $z = 0$ at the centers of the horizontal tubes.

The profile of this junction is shown below. The blue lines show the intersections with two connecting tubes, showing that the tubes never touch each other directly. The red line shows the closest approach to the other color, assuming $r = 0.3s$.




Using sloping tubes makes it easier to drain the unit and increases the minimum distance from the junction to the nearest tube of the other color. However sloping tubes add more fluid for a given value of $z$.

From the point where $z$ intercepts the tube to radius $R(z)$ is an arch through which fluid is rising at slope angle $\phi$. Projecting that arch perpendicular to the flow foreshortens it by $\sin\phi$. The radius must satisfy the set of equations

$$\pi R^2(\theta) = \pi r^2 + 4r^2\left(\theta + \frac{\pi}{2} + \frac{\sin 2\theta}{2}\right) + 4R^2(\theta)\left(\alpha - \frac{\sin 2\alpha}{2}\right)\sin\phi$$
$$R(\theta)\sin\alpha = r\cos\theta$$

for some unknown $\alpha$ This has no closed form solution, but can be with successive approximations.

The intersection can be modeled parametrically using

$$z(\theta) = r\sin\theta\sec\phi - \left(\sqrt{R^2(\theta) - r^2\cos^2\theta} - \sqrt{R^2(0) - r^2}\right)\tan\phi$$

for slope angle $\phi$. The $\sec\phi$ term is the projection of the cylinder onto a vertical plane, the $\tan\phi$ term is the change due to the varying radius (using Pythagoras) and the $\sqrt{R^2(0) - r^2)}$ term forces $z(0) = 0$. Also, $R(0) > \sqrt{3}\, r$ and needs to be precomputed.

The largest possible slope can be found using

$$\phi = \sin^{-1}\left(\frac{\sqrt{5}-1}{2}\right) \approx 38.2^\circ$$

and the vertical distance for the tube centers from the start to end is given by

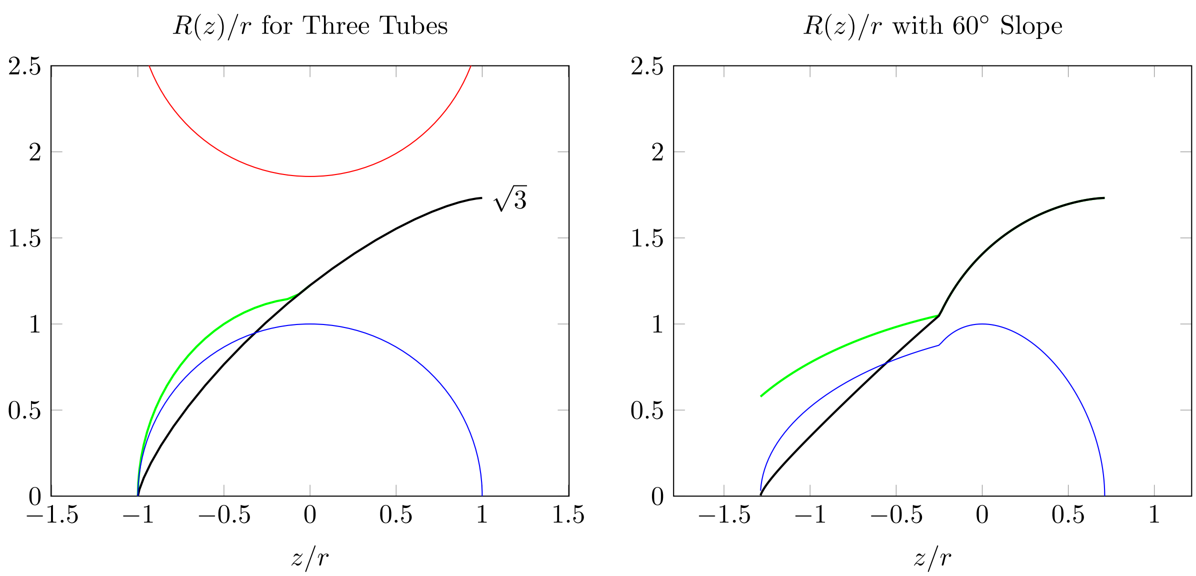
$$\Delta z = (s - R(0)) \tan\phi$$

which is constant across the layer. Technically, one could use any slope and change the angle when you reach the junction.

At the junction of three tubes (hexagonal grid) the minimum radius $R(z)$ is given by

$$R(z) = r\sqrt{3\left(\frac{\theta}{\pi} + \frac{1}{2} + \frac{\sin 2\theta}{2\pi}\right)}\ \Bigg|_{\theta=\sin^{-1}(z/r)}$$

which is shown below.




This time, the horizontal tubes **will** intersect each other, even at 120° apart. It is better to use the ellipsoid

$$R(z) = \frac{2}{\sqrt{3}}\, r\cos\theta\ \Bigg|_{\theta=\sin^{-1}(z/r)}$$

shown in green. This shape will intersect the tubes at precisely the same point they intersect each other, since $R(\theta)\sin 60^\circ = r\cos\theta$. More than just a conceptual aid, it makes the math simpler.

Adding slope causes a dimple at the bottom, violating the model of a contiguous column of fluid. At the very least, one needs to handle the outer and inner surfaces separately. An outer surface using

$$R(z) = r\sqrt{\frac{1}{3} + \cos^2\theta}\ \Bigg|_{\theta=\sin^{-1}(z/r)}$$

will guarantee that $dz/d\theta \geq 0$ for $-90° \leq \theta \leq 0$ for any slope $\phi \geq 0$, since

$$z = r\sin\theta\sec\phi - \left(\frac{r}{\sqrt{3}} - \sqrt{R^2(0) - r^2}\right)\tan\phi$$

when using the new shape.

The largest possible slope the junction can handle is given by

$$\phi = \sin^{-1}\left(\sqrt{3}/2\right) = 60° \quad .$$

One can use the minimum radius (area) calculations to determine the maximum size of an inner surface (dimple), but streamlining is outside the scope of this paper.

## Final Notes

For a heat exchanger, the last layer should expand the radius by $\sqrt{2}$ or $2/\sqrt{3}$ to increase the surface area and maintain the thickness/diameter ratio. It would probably be best to divide the exchanger into three sections, with a fractal manifold on each end and a variable length section with constant radius tubes in the middle.

A burner should end with expansion cones to minimize the contact between the flame and metal walls. I doubt one would need additional turbulence.