Appendix D
Acoustic Wave Equation - Solutions

We can write the general acoustic wave equation as

$$\nabla^2 p = \left(\frac{1}{c^2}\right)\frac{\partial^2 p}{\partial t^2} \tag{D1}$$

where $p \doteq P - P_0$ and c is the speed of sound. As shown in Appendix C this is actually an approximation, but is adequate for our purposes. In the frequency domain this simplifies to

$$\nabla^2 p_\omega = \frac{-\omega^2}{c^2} p_\omega \tag{D2}$$

where

$$p(t) = p_\omega e^{-j\omega t}$$

for outgoing waves and p_ω is generally complex.

Plane Waves

In rectangular coordinates, the gradient is given by

$$\vec{\nabla} = \vec{x}\frac{\partial}{\partial x} + \vec{y}\frac{\partial}{\partial y} + \vec{z}\frac{\partial}{\partial z}$$

and the Laplacian is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

from which one can derive the general plane wave solution

$$p_\omega = A_\omega e^{j(k_x x + k_y y + k_z z)} \tag{D3}$$

for some constant A_ω where

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \quad . \tag{D4}$$

Applying $(C2)$ to $(D3)$ one can show, for example,

$$jk_x p_\omega = j\omega\rho v_x(\omega)$$

and repeating for y and z gives us

$$\begin{aligned} v_x(\omega) &= k_x \frac{p_\omega}{\omega\rho} \\ v_y(\omega) &= k_y \frac{p_\omega}{\omega\rho} \\ v_z(\omega) &= k_z \frac{p_\omega}{\omega\rho} \end{aligned}$$

and therefore from $(D4)$ we can write

$$v_\omega \doteq \sqrt{v_x^2 + v_y^2 + v_z^2} = \frac{1}{c\rho} p_\omega \tag{D5}$$

which is equivalent to the approximation in $(C9)$.

Cylindrical Waves

In cylindrical coordinates, the gradient is given by

$$\vec{\nabla} = \vec{r}\frac{\partial}{\partial r} + \vec{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \vec{z}\frac{\partial}{\partial z}$$

and the Laplacian is given by

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

where $x = r\cos\phi$ and $y = r\sin\phi$. The axially symmetrical solution is given by

$$p_\omega = B_\omega J_0(k_r r)e^{jk_z z} \tag{D6}$$

for some constant B_ω, where J_0 is a Bessel function and

$$k_r^2 + k_z^2 = \frac{\omega^2}{c^2} \quad . \tag{D7}$$

Applying $(C2)$ to $(D2)$ we get

$$v_z(\omega) = \frac{k_z}{\omega\rho}p_\omega$$

as before, but the radial component is given by

$$v_r(\omega) = \frac{k_r}{j\omega\rho}B_\omega J_1(k_r r)e^{jk_z z} \quad . \tag{D8}$$

While v_z is in phase with p_ω, v_r is out of phase in both the r and z directions.

Spherical Waves

In spherical coordinates, the gradient is given by

$$\vec{\nabla} = \vec{r}\frac{\partial}{\partial r} + \vec{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \vec{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

and the Laplacian is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

where

$$\begin{aligned} x &= r\sin\theta\cos\phi \quad , \\ y &= r\sin\theta\sin\phi \quad \text{and} \\ z &= r\cos\theta \quad . \end{aligned}$$

The spherically symmetrical solution is given by

$$p_\omega = C_\omega \frac{1}{r}e^{j\omega r/c} \tag{D9}$$

and the axially symmetrical solution is given by

$$p_\omega = C_\omega h_m(\omega r/c)P_m(\cos\theta) \tag{D10}$$

for some constant C_ω, where P_m is a Legendre polynomial and h_m is a spherical Hankel function.

Applying $(C2)$ to $(D9)$ we get

$$\left(\frac{j\omega}{c} - \frac{1}{r}\right) p_\omega = j\omega\rho v_r(\omega)$$

which means that

$$v_r(\omega) \to \frac{p_\omega}{\rho c} \quad \text{as} \quad r \to \infty \tag{D11}$$

and

$$v_r(\omega) \to \frac{-p_\omega}{j\omega\rho r} \quad \text{as} \quad r \to 0 \quad . \tag{D12}$$

This means that v_r and p_ω are in phase in the far field but out of phase in the near field ($r \ll c/\omega$).

Bessel Functions, etc.

Bessel functions are defined as solutions to the differential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \left(1 - \frac{m^2}{x^2}\right)\right) J_m(x) = 0$$

and can be computed using the power series

$$J_n(x) = \frac{(\frac{x}{2})^n}{n!} - \frac{(\frac{x}{2})^{n+2}}{1!(n+1)!} + \frac{(\frac{x}{2})^{n+4}}{2!(n+2)!} - \dots \quad .$$

It is sometimes useful to approximate them using

$$J_m(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - (2m+1)\frac{\pi}{4}\right) \quad \text{as} \quad x \to \infty.$$

Legendre polynomials are simply polynomials of order m which satisfy the orthogonality condition

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \begin{cases} 0 & \text{for } m \neq n \\ \frac{2}{2m+1} & \text{for } m = n \end{cases}$$

starting with $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = x^2 - 1/3$.

Spherical Hankel functions are solutions to the differential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} + \left(1 + \frac{m(m+1)}{x^2}\right)\right) h_m(x) = 0$$

and have the form

$$h_m(x) = \frac{e^{jx}}{j^{m+1}x} \sum_{i=0}^{m} \frac{(m+i)!}{i!(m-i)!} \left(\frac{i}{2x}\right)^i \quad .$$

It is sometimes useful to approximate this function using

$$h_m(x) \to \frac{(2m)!}{jm!x} \left(\frac{1}{2x}\right)^m \quad \text{as} \quad x \to 0$$

or

$$h_m(x) \to \frac{e^{jx}}{j^{m+1}x} \quad \text{as} \quad x \to \infty.$$

2D Fourier Transforms

The two dimensional Fourier transform of function $f(x,y)$ is given by

$$F(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-jk_x x}e^{-jk_y y}\,dx\,dy \tag{D13}$$

where k_x and k_y represent spacial frequencies. The corresponding inverse 2D Fourier transform is of the form

$$f(x,y) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_x,k_y)e^{j(k_x x+k_y y)}\,dk_x\,dk_y \quad .$$

When $f(x,y)$ represents a wave function at $z=0$, one can represent the wave function over all space as a collection of plane waves of the form

$$f(x,y,z) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_x,k_y)e^{j(k_x x+k_y y+k_z z)}\,dk_x\,dk_y \tag{D14}$$

where k_z^2 satisfies (D4). Note, when

$$k_x^2 + k_y^2 > \left(\frac{\omega}{c}\right)^2$$

the resulting wave function decays exponentially with z (an effenescent wave).

In cylindrical coordinates we define radial frequency k_r and angle θ such that

$$k_x = k_r\cos\theta \quad \text{and} \quad k_y = k_r\sin\theta \quad .$$

The 2D Fourier transform in this domain is given by

$$F(k_r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} f(r,\theta)e^{-jk_r r\cos(\theta-\phi)}r\,dr\,d\phi$$

since

$$\cos(\theta-\phi) = \cos\phi\cos\theta + \sin\phi\sin\theta \quad .$$

When f is radially symmetrical, one can perform the integration over θ to obtain

$$F(k_r) = \int_0^{\infty} f(r)J_0(k_r r)r\,dr \tag{D15}$$

since

$$J_m(x) = \frac{1}{2\pi j^m}\int_0^{2\pi} e^{jx\cos\theta}\cos(m\theta)\,d\theta \quad .$$

Note that $(D15)$ is a zero'th order Hankel transform, and its inverse transform is simply

$$f(r) = \int_0^{\infty} F(k_r)J_0(k_r r)k_r\,dk_r \quad . \tag{D16}$$

Once again, if $f(r)$ represents the wave function at $z=0$, the complete solution is given by

$$f(r,z) = \int_0^{\infty} k_r F(k_r)J_0(k_r r)e^{jk_z z}\,dk_r \tag{D17}$$

where k_z satisfies $(D7)$.

Far Field Approximation

One can write the general spherical wave solution as

$$p_\omega \to C_\omega(\theta,\phi)\frac{1}{r}e^{j\omega r/c} \quad \text{as} \quad r \to \infty \tag{D18}$$

since the relative amplitude as a function of angle asymptotically approaches a constant. (The Legendre polynomial decomposition is only important in the near field.)

Further, one can treat the solution at $z = 0$ as a diffraction grating or a phased array antenna and determine C_ω by the extent to which the "sources" are in phase. Taking the origin as our base point, the relative delay from some point $(x, y, 0)$ to a distant point in the (θ, ϕ) direction is given by

$$t(x,y) = -\frac{x}{c}\sin\theta\cos\phi - \frac{y}{c}\sin\theta\sin\phi$$

and the resulting phased contributions for source function $f(x, y)$ are given by

$$C_\omega(\theta,\phi) = \alpha\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{j\omega t(x,y)}\,dx\,dy$$

for some scale factor α. Comparing this to $(D13)$ we see that

$$C_\omega(\theta,\phi) = \alpha F(k_x,k_y) \tag{D19}$$

when

$$k_x = \frac{\omega}{c}\sin\theta\cos\phi \quad \text{and} \quad k_y = \frac{\omega}{c}\sin\theta\sin\phi \quad .$$

Similarly, the axially symmetrical solution is given by

$$C_\omega(\theta) = \alpha F(k_r) \tag{D20}$$

where

$$k_r = \frac{\omega}{c}\sin\theta \quad .$$

To solve for scale factor α one can reduce the problem to a point source. Any monopole source will produce spherical waves if you make it small enough. The net flow of air from the source equals the net flow from a small sphere which contains the source, but the velocity is reduced by the ratio of the two surface areas. You simply equate the resulting solution in spherical coordinates to the limit of the Fourier transform as the size goes to zero.