Appendix C
Acoustic Wave Equation - Derivation

Define pressure P, density ρ and average particle motion $\vec{v}$ fields over a range of space. The divergence of mass flow $\rho\vec{v}$ represents the net flow of mass out of a infinitesimaly small region, reducing the mass within that region. This can be expressed mathematically as

$$\vec{\nabla} \cdot \rho\vec{v} = -\partial\rho\partial t \quad . \tag{C1}$$

A pressure gradient will cause air to flow in the opposite direction (the gradient vector points toward increasing pressure). From Newton's second law we can write this as

$$\vec{\nabla} P = -\frac{\partial}{\partial t}\rho\vec{v} \quad . \tag{C2}$$

Taking the divergence of the pressure gradient we get

$$\begin{aligned} \vec{\nabla} \cdot \vec{\nabla} P &= -\vec{\nabla} \cdot \frac{\partial}{\partial t}\rho\vec{v} \\ &= -\frac{\partial}{\partial t}(\vec{\nabla} \cdot \rho\vec{v}) \end{aligned}$$

and therefore

$$\nabla^2 P = \frac{\partial^2 \rho}{\partial t^2} \quad . \tag{C3}$$

Isothermal Wave Equation

From the ideal gas law we have

$$P = \rho k T \tag{C4}$$

where T is temperature in Kelvin and k is Boltzman's constant. Substituting for density and ignoring variations in temperature we obtain the isothermal wave equation

$$\nabla^2 P = \left(\frac{1}{kT}\right)\frac{\partial^2 P}{\partial t^2} \quad . \tag{C5}$$

Comparing this with the general accoustic wave equation ($D1$) reveals that the speed of sound is given by $c = \sqrt{kT}$.

Adiabatic Wave Equation

The adiabatic solution is found by using the work done to compress volume V by $-dV$ to warm up the air by dT, or

$$\mu M\, dT = -P\, dV = -\frac{MkT}{V}\, dV$$

where μ is the specific heat for air. This yields the differential equation

$$\frac{dT}{dV} = -(k/\mu)\frac{T}{V}$$

which has the solution

$$T = aV^{-k/\mu}$$

for some constant a. If $T = T_0$ when $V = V_0$, we can rewrite this as

$$\frac{T}{T_0} = (VVV_0)^{-k/\mu} \quad .$$

Substituting this back into the ideal gas law (C4), one can generate the relationships

$$\frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{1+\mu/k} = \left(\frac{\rho}{\rho_0}\right)^{1+k/\mu} \tag{C6}$$

and therefore

$$\frac{\rho}{\rho_0} = \left(\frac{P}{P_0}\right)^{\gamma} \quad \text{where} \quad \gamma = \frac{1}{1+k/\mu} \quad .$$

Differentiating this twice with respect to time gives us

$$\frac{1}{\rho_0}\frac{\partial^2 \rho}{\partial t^2} = \frac{\gamma}{P_0}\left(\frac{P}{P_0}\right)^{\gamma-1}\frac{\partial^2 P}{\partial t^2} + \frac{\gamma(\gamma-1)}{P_0^2}\left(\frac{P}{P_0}\right)^{\gamma-2}\left(\frac{\partial P}{\partial t}\right)^2$$

which simplifies to

$$\frac{\partial^2 \rho}{\partial t^2} = \left(\frac{\gamma}{kT}\right)\frac{\partial^2 P}{\partial t^2} + \left(\frac{\gamma^2}{\mu T}\right)\frac{1}{P}\left(\frac{\partial P}{\partial t}\right)^2 \tag{C7}$$

noting that $T/T_0 = (P/P_0)^{1-\gamma}$ is a variable.

When the variations in temperature and pressure are small compared to their actual values at any given time, we can drop the term using $(dP/dt)^2/P$ and replace T with T_0, simplifying the wave equation $(C3)$ to the form

$$\nabla^2 P = \left(\frac{\gamma}{kT_0}\right)\frac{\partial^2 P}{\partial t^2} \tag{C8}$$

which has a linear solution. In fact, the only difference is that now the speed of sound is given by $c = \sqrt{kT_0/\gamma}$.

Particle Velocity

When the variations in density are small, from $(C2)$ one can write

$$\vec{\nabla} P \approx -\rho_0 \frac{\partial \vec{v}}{\partial t} \quad .$$

For a plane wave of the form

$$P - P_0 = p_\omega e^{j\omega(t-x/c)} \quad \text{and} \quad v = v_\omega e^{j\omega(t-x/c)}$$

we can obtain

$$\frac{-j\omega}{c} p_\omega \approx -\rho_0(j\omega v_\omega)$$

and therefore

$$v_\omega \approx \frac{p_\omega}{\rho_0 c} \quad . \tag{C9}$$

A more rigorous handling of the adiabatic effects yields the nonlinear relationship

$$P - P_0 = \frac{\rho c v}{1 - v/c}$$

which differs from $(C9)$ only when the particle velocity approaches the speed of sound (shock waves).