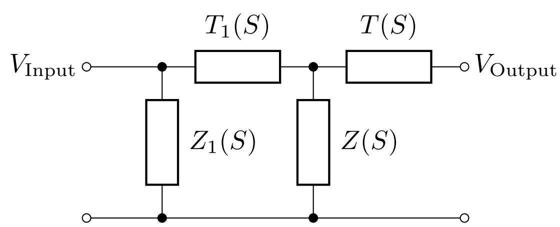
Appendix B
Derivation of Transfer Functions



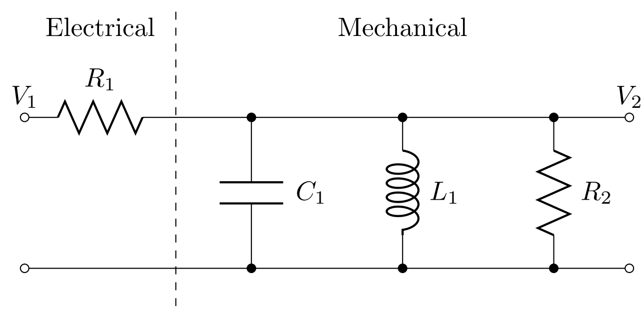
Block Model

We will ignore the presence of the air loading $Z(S)$ and coupling $T(S)$ and derive the transfer function $T_1(S)$ and input impedance $Z_1(S)$ based on the remaining components. Given $T_1(S)$ and $Z_1(S)$, one can obtain the total transfer function using

$$V_{\text{Output}} = T_1(S)T(S)\,V_{\text{Input}} \tag{B1}$$

and the total impedance is given by

$$V_{\text{Input}} = \frac{Z_1(S)T_1(S)Z(S)}{Z_1(S) + T_1(S)Z(S)}\,I_{\text{Input}} \quad . \tag{B2}$$



Infinite Baffle

Summing the loop currents flowing out of the node with voltage V_2 we get

$$\frac{V_2 - V_1}{R_1} + SCV_2 + \frac{V_2}{SL} + \frac{V_1}{R_2} = 0$$

and therefore

$$\frac{V_1}{R_1} = \frac{S^2CLR + SL + R}{SLR}V_2$$

where $R = R_1R_2 = (R_1 + R_2)$. Transfer function $T_1(S)$ is given by

$$T_1(S) = \frac{V_2}{V_1} = \left(\frac{R_2}{R_1 + R_2}\right)\frac{SL}{S^2CLR + SL + R} \quad . \tag{B3}$$

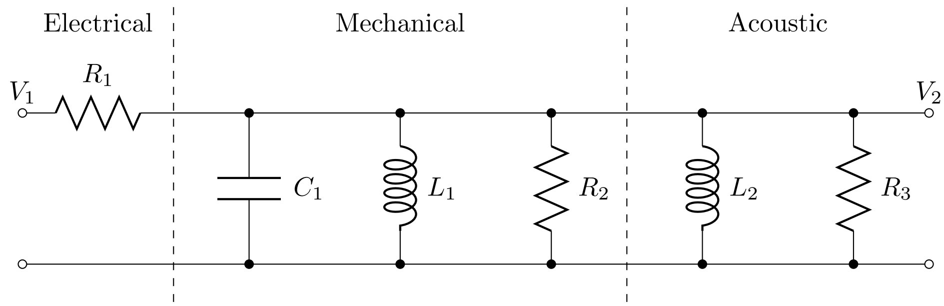
The input current passes through R_1 and satisfies

$$I_1 = \frac{V_1 - V_2}{R_1} = \frac{1 - T_1(S)}{R_1} V_1(S)$$

and the input impedance is then given by

$$Z_1(S) = \frac{V_1}{I_1} = \frac{R_1}{1 - T_1(S)} \quad . \qquad (B4)$$

Note that one should replace the air loading impedance $Z(S)$ with $\frac{1}{2}Z(S)$ when computing the total impedance, since the air loading applies to both sides of the speaker.



Simple Enclosure

If one replace inductors L_1 and L_2 with $L = L_1 L_2/(L_1 + L_2)$ and resistors R_2 and R_3 with $R_{23} = R_2 R_3/(R_2 + R_3)$, the resulting circuit is almost identical to that of the infinite baffle. Specifically, the transfer function is now given by

$$T_1(S) = \frac{V_2}{V_1} = \left(\frac{R_{23}}{R_1 + R_{23}}\right) \frac{SL}{S^2 CLR + SL + R} \qquad (B5)$$

where $R = R_1 R_{23}/(R_1 + R_{23})$ and therefore

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

and the input impedance $Z_1(S)$ is still given by

$$Z_1(S) = \frac{V_1}{I_1} = \frac{R}{1 - T_1(S)} \quad . \qquad (B6)$$

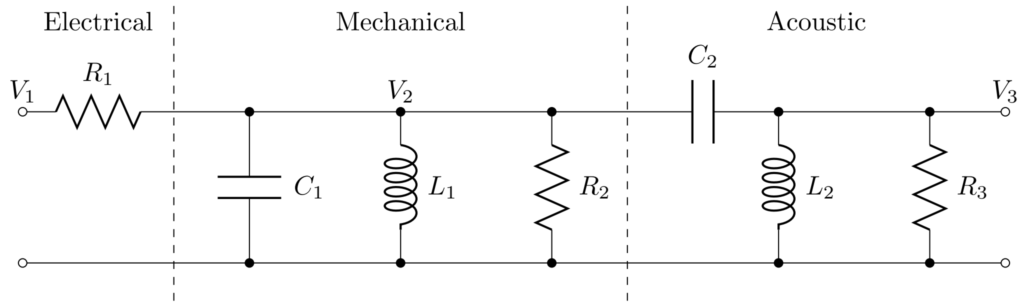
The ideal transfer function has a maximally smooth (Butterworth) response, for which the second order high pass version has the form

$$\frac{S^2}{S^2 + 1.414\omega_0 S + \omega_0^2}$$

where $\omega_0 = 2\pi f_0$ is the (-3dB) cutoff frequency[1, p. 539]. Since air coupling $T(S)$ contains the addition power of S needed, $T_1(S)T(S)$ has a Butterworth response when

$$\omega_0 = \frac{0.707}{RC} \quad \text{and} \quad L = 2R^2 C \quad .$$

Adding sound absorbing materials reduces R and allows one to use a smaller enclosure, although with a higher cutoff frequency.



Base Reflex

Summing the loop currents flowing out of the node with voltage V_3 we get

$$(V_3 - V_2)SC_2 + \frac{V_3}{R_3} + \frac{V_3}{SL_2} = 0 \quad ,$$

$$SC_2V_2 = \frac{S^2C_2L_2R_3 + SL_2 + R_3}{SL_2R_3}V_3 \quad ,$$

$$V_2 = \frac{S^2C_2L_2R_3 + SL_2 + R_3}{S^2L_2C_2R_3}V_3 \quad ,$$

and therefore

$$V_2 - V_3 = \frac{SL_2 + R_3}{S^2L_2C_2R_3}V_3 \quad .$$

Summing the loop currents flowing out of the node with voltage V_2 we get

$$\frac{V_2 - V_1}{R_1} + SC_1V_2 + \frac{V_2}{SL_1} + \frac{V_2}{R_2} + (V_2 - V_3)SC_2 = 0$$

and therefore

$$\frac{V_1}{R_1} = \frac{S^2C_1L_1R_{12} + SL_1 + R_{12}}{SL_1R_{12}}V_2 + SC_2(V_2 - V_3)$$

where $R_{12} = R_1R_2/(R_1 + R_2)$. Sustituting for V_2 and $(V_2 - V_3)$ as derived above, we get

$$\frac{V_1}{R_1} = \left(\frac{S^2C_1L_1R_{12} + SL_1 + R_{12}}{SL_1R_{12}}\right)\left(\frac{S^2C_2L_2R_3 + SL_2 + R_3}{S^2C_2L_2R_3}\right)V_3 + \frac{SL_2 + R_3}{SL_2R_3}V_3$$

and therefore

$$\frac{V_1}{R_1} = \frac{f(S)}{S^3C_2L_1L_2R_{12}R_3}V_3$$

where

$$\begin{aligned} f(S) \quad = \quad & S^4C_1C_2L_1L_2R_{12}R_3 + S^3L_1L_2(C_1R_{12} + C_2R_3 + C_2R_{12}) \\ & +S^2((C_1L_1 + C_2L_2 + C_2L_1)R_{12}R_3 + L_1L_2) \\ & +S(L_1R_3 + L_2R_{12}) + R_{12}R_3 \quad . \end{aligned}$$

Finally, the transfer function is given by

$$T_1(S) = \frac{V_3}{V_1} = \frac{S^3C_2L_1L_2R_{12}R_3}{R_1f(S)} \qquad (B7)$$

and V_2 is given by

$$\begin{aligned} V_2 &= \left(\frac{S^2 C_2 L_2 R_3 + S L_2 + R_3}{S^2 L_2 C_2 R_3}\right) T_1(S) V_1 \\ &= S L_1 R_{12} \left(\frac{S^2 C_2 L_2 R_3 + S L_2 + R_3}{R_1 f(S)}\right) V_1 \quad . \end{aligned}$$

Substituting this back into

$$I_1 = \frac{V_1 - V_2}{R_1} = \frac{V_1}{R_1}\left(1 - \frac{V_2}{V_1}\right)$$

gives us

$$I_1 = \frac{R_{12} g(S)}{R_1 R_2 f(S)} V_1$$

where

$$\begin{aligned} g(S) &= \frac{R_{12}}{R_1 R_2}\left(R_1 f(S) - S^3 C_2 L_1 L_2 R_{12} R_3 - S^2 L_1 L_2 R_{12} - S L_1 R_{12} R_3\right) \\ &= S^4 C_1 C_2 L_1 L_2 R_2 R_3 + S^3 L_1 L_2 (C_1 R_2 + C_2 R_2 + C_2 R_3) \\ &\quad + S^2((C_1 L_1 + C_2 L_2 + C_2 L_1) R_2 R_3 + L_1 L_2) \\ &\quad + S(L_2 R_1 + L_1 R_3) + R_2 R_3 \end{aligned}$$

since

$$R_1 - R_{12} = \frac{R_1^2}{R_1 + R_2} = R_1 R_{12}/R_2 \quad .$$

The input impedance is given by

$$Z_1(S) = \frac{V_1}{I_1} = (R_1 + R_2)\frac{f(S)}{g(S)} \quad . \qquad (B8)$$

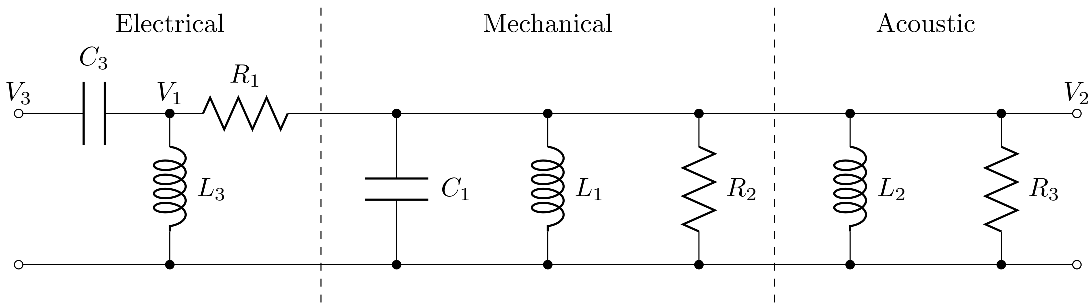
Given that C_1, L_1, R_1 and R_2 are characteristics of the speaker itself, we would like to obtain values for C_2, L_2 and R_3 to achieve the fourth order Butterworth polynomial

$$\left(\frac{S}{\omega_0}\right)^4 + 2.61\left(\frac{S}{\omega_0}\right)^3 + 3.41\left(\frac{S}{\omega_0}\right)^2 + 2.61\left(\frac{S}{\omega_0}\right) + 1 = \frac{f(S)}{R_{12} R_3}$$

where the cutoff frequency $\omega_0 = 2\pi f_0$ is also uniquely determined [1, p 539]. These values can be found by solving the following set of non-linear equations:

$$\begin{aligned} C_1 C_1 L_1 L_2 \omega^4 &= 1 \\ L_1 L_2 \left(\frac{C_1}{R_3} + \frac{C_2}{R_3} + \frac{C_2}{R_{12}}\right)\omega^3 &= 2.61 \\ \left(C_1 L_1 + C_1 L_2 + C_2 L_1 + \frac{L_1 L_2}{R_{12} R_3}\right)\omega^2 &= 3.41 \\ \left(\frac{L_1}{R_{12}} + \frac{L_2}{r_3}\right)\omega &= 2.61 \end{aligned}$$

Unfortunately, there is no closed form solution for this problem. One can approach a solution for a particular set of parameters using nonlinear optimization (see Appendix F).



Crossover Filter

Summing the loop currents flowing out of the node with voltage V_1 we get

$$(V_1 - V_3)SC_3 + \frac{V_1 - V_2}{R_1} + \frac{V_1}{SL_3} = 0$$

and therefore

$$SC_3V_3 = \frac{S^2C_3L_3R_1 + SL_3 + R_1}{SL_3R_1}V_1 - \frac{V_2}{R_1} \quad .$$

One can substitute for V_1 using $(B5)$ to obtain

$$\begin{aligned} SC_3V_3 &= \frac{(S^2C_3L_3R_1 + SL_3 + R_1)(S^2CLR + SL + R)}{S^2L_3LR}V_2 - \frac{V_2}{R_1} \\ &= \frac{p(S)}{S^2L_3LR} \end{aligned}$$

where

$$\begin{aligned} p(S) &= S^4C_3CL_3LR_1R + S^3L_3L(C_3R_1 + CR) \\ &\quad + S^2(C_3L_3R_1R + CLRR_1 + L_3LR/R_{23}) \\ &\quad + S(L_3R + LR_3) + R_3R \end{aligned}$$

since

$$1 - \frac{R}{R_1} = \frac{R}{R_{23}} \quad .$$

The transfer function is given by

$$T_1(S) = \frac{V_2}{V_3} = \frac{S^3C_2L_1L_2R}{p(S)} \tag{B9}$$

and using $(B5)$ again we get

$$V_1 = R_1\left(\frac{S^2CLR + SL + R}{SLR}\right)T_1(S)V_3 \quad .$$

Substituting this back into

$$I_3 = SC_3(V_3 - V_1)$$

together with $(B9)$ gives us

$$I_3 = \frac{C_3}{LRp(S)}\left(SLRp(S) - S^3C_3L_3LR_1R(S^2CLR + SL + R)\right)V_3$$

or

$$I_3 = \frac{C_3 q(S)}{p(S)} V_3$$

where

$$q(S) = S^4 C L_3 L R + S^3 L R (C R_1 + L_3/R_{23}) + S^2 (L_3 R + L R_1) + S R_3 R \quad .$$

The input impedance is given by

$$Z_1(S) = \frac{V_3}{I_3} = \frac{f(S)}{C_3 g(S)} \quad . \qquad (B10)$$

Once again, there is no closed form solution for a fourth order Butterworth.

For active filters, however, a simpler approach is possible. A fourth order high pass Butterworth filter can be decomposed as

$$\left(\frac{S^2}{S^2 + 0.766\omega_0 S + \omega_0^2} \right) \left(\frac{S^2}{S^2 + 1.898\omega_0 S + \omega_0^2} \right)$$

so that one can implement one of the above using the speaker itself and the rest electronically. The solutions for a simple enclosure will be given by

$$\omega_0 = \frac{0.383}{RC} \qquad\qquad \omega_0 = \frac{0.949}{RC}$$

$$L = 6.82 R^2 C \qquad\qquad L = 1.11 R^2 C$$

depending on which part is implemented. Obviously the left solution gives a much lower cutoff frequency, but requires a much larger value for L. Since L can never exceed its value from the infinite baffle, you may sometimes have to go with the second solution.

References

[1] Wallace L. Cassell, **Linear Electric Circuits**, LCCN 64-17134