Appendix G
Exponential Horns

Consider a horn with cross-section area function $A(z)$. The net mass flow into and out of a slice of width Δz equals the change in mass per time within that volume, which can be written as

$$\rho(z)v(z)A(z) - \rho(z+\Delta z)v(z+\Delta z)A(z+\Delta z) = A(z)\Delta z\frac{\partial \rho}{\partial t}$$

where ρ is the density and v is the particle velocity in the z direction. Dividing both sides by Δz and taking the limit as $\Delta z \to 0$ gives us

$$-\frac{\partial}{\partial z}\rho v A(z) = A(z)\frac{\partial \rho}{\partial t} \tag{G1}$$

This differs from equation $(C1)$ since particles cannot travel through the horn walls. Since the walls do not alter the effect of pressure, equation $(C2)$ remains the same. In fact one can simplify it as

$$\frac{\partial P}{\partial z} = -\frac{\partial \rho v}{\partial t} \tag{G2}$$

since we are dealing with simple plane waves in z.

Taking another derivative of $(G1)$ w.r.t. time gives us

$$\begin{aligned} A(z)\frac{\partial^2 \rho}{\partial t^2} &= -\frac{\partial}{\partial z}A(z)\frac{\partial \rho v}{\partial t} \\ &= \frac{\partial}{\partial z}A(z)\frac{\partial P}{\partial z} \quad . \end{aligned}$$

Finally, using the linearity assumption

$$P - P_0 = (\rho - \rho_0)c^2$$

where c is the speed of sound, we get the 1 dimensional wave equation

$$\frac{\partial^2 P}{\partial t^2} = \left(\frac{c^2}{A(z)}\right)\frac{\partial}{\partial z}A(z)\frac{\partial P}{\partial z} \tag{G3}$$

for any horn with a continuous area function.

For an exponential horn, the area function is given by

$$A(z) = A_0 e^{2bz}$$

for some constant b (note that the diameter increases by only e^{bz}). Substituting this back into $(G3)$ gives us the wave equation

$$\frac{\partial^2 P}{\partial t^2} = c^2\left(2b\frac{\partial P}{\partial z} + \frac{\partial^2 P}{\partial z^2}\right) \quad . \tag{G4}$$

This has the general solution

$$P - P_0 = p_\omega e^{j\omega t + \beta z} \tag{G5}$$

where β satisfies the quadratic equation

$$\beta^2 + 2b\beta + \frac{\omega^2}{c^2} = 0 \quad ,$$

which has the solution

$$\beta = -b \pm \sqrt{b^2 - \omega^2/c^2} \quad . \tag{G6}$$

Since the variations in ρ are much smaller than ρ_0, from $(G2)$ we have

$$\frac{dP}{dz} \approx -\rho_0 \frac{dv}{dt}$$

which has the solution

$$v = v_\omega e^{j\omega t + \beta z} \tag{G7}$$

where

$$v_\omega \approx \frac{j\beta}{\omega \rho_0} p_\omega \quad . \tag{G8}$$

When $\omega > bc$, the solution can be written as

$$v = v_\omega e^{-bz} e^{j(\omega t \pm kz)} \tag{G9}$$

where

$$k = \sqrt{\omega^2/c^2 - b^2} \quad . \tag{G10}$$

While both particle velocity and pressure decrease as z increases, far field sound volume is proportional to velocity times the opening area, or

$$v_\omega e^{-bz} A(z) = v_\omega A_0 e^{bz}$$

for a horn of length z. Note also that the phase velocity within the horn,

$$\frac{\omega}{k} = c/\sqrt{1 - (bc/\omega)^2} \quad , \tag{G11}$$

varies with frequency, especially near $\omega = bc$ where it becomes infinite.

When $\omega < bc$, the two solutions have different rates of exponential decrease. In the limit as the frequency goes to zero, "outgoing waves" decrease by e^{-2bz} and "incoming waves" remains constant (no amplification in either direction). To handle lower frequencies one must reduce the exponent b, which results in a longer horn.

Coupling

In general there will exist two plane wave solutions (foreward and backward) inside the horn, and the pressure and velocity at any given point is the sum of both components. When $\omega > bc$, from $(G6)$, $(G8)$ and $(G10)$ we see that the foreward component satisifes

$$p_f(z) = \frac{j\omega\rho_0}{b + jk} v_f(z)$$

while the backward component satisfies

$$p_b(z) = \frac{j\omega\rho_0}{b - jk} v_b(z)$$

and therefore

$$p_f(z) + p_b(z) = \frac{j\omega\rho_0}{b^2 + k^2} \big((b - jk) v_f(z) + (b + jk) v_b(z) \big) \quad . \tag{G12}$$

It should also be noted that, from $(G10)$,

$$b^2 + k^2 = \frac{\omega^2}{c^2} \quad .$$

The acoustic impedance at the mouth of a horn of length ℓ satisfies

$$Z(\omega) = A(\ell)\frac{v_f(\ell) + v_b(\ell)}{\big(p_f(\ell) + p_b(\ell)\big)}$$

and therefore, using ($G12$),

$$\frac{Z(\omega)}{A(\ell)} = \left(\frac{-j\omega}{c^2\rho_0}\right)\frac{v_f(\ell) + v_b(\ell)}{(b - jk)v_f(\ell) + (b + jk)v_b(\ell)}$$

from which we can derive

$$\frac{v_b(\ell)}{v_f(\ell)} = -\frac{j\omega A(\ell) + Z(\omega)c^2\rho_0(b - jk)}{j\omega A(\ell) + Z(\omega)c^2\rho_0(b + jk)} \quad . \tag{G13}$$

Since a power amplifier will supply the desired voltage to the speaker regardless of the impedance, one can regard diaphragm velocity, $v(0)$, as a known quantity. From ($G13$) and

$$\frac{v_b(0)}{v_f(0)} = e^{-jk2\ell}\frac{v_b(\ell)}{v_f(\ell)}$$

we can derive the solutions

$$v_f(0) = \frac{v(0)}{2}e^{jk\ell}\frac{k - j\left(b + \frac{j\omega A(\ell)}{Z(\omega)c^2\rho_0}\right)}{k\cos(k\ell) + \left(b + \frac{j\omega A(\ell)}{Z(\omega)c^2\rho_0}\right)\sin(k\ell)} \tag{G14}$$

and

$$v_b(0) = \frac{v(0)}{2}e^{-jk\ell}\frac{k + j\left(b + \frac{j\omega A(\ell)}{Z(\omega)c^2\rho_0}\right)}{k\cos(k\ell) + \left(b + \frac{j\omega A(\ell)}{Z(\omega)c^2\rho_0}\right)\sin(k\ell)} \quad . \tag{G15}$$

From these one can easily show that the transmission gain through the horn is given by

$$\begin{aligned} T_h(\omega) &= \frac{A(\ell)v(\ell)}{A_0 v(0)} \\ &= e^{b\ell}\left(\frac{v_f(0)}{v(0)}e^{-jk\ell} + \frac{v_b(0)}{v(0)}e^{jk\ell}\right) \\ &= \frac{e^{b\ell}}{\cos(k\ell) + \left(b + \frac{j\omega A(\ell)}{Z(\omega)c^2\rho_0}\right)\frac{\sin(k\ell)}{k}} \end{aligned} \tag{G16}$$

where

$$\lim_{\omega\to bc} T_h(\omega) = \frac{e^{b\ell}}{1 + b\ell\left(1 + \frac{jA(\ell)}{Z(bc)c\rho_0}\right)} \quad .$$

Similarly, the acoustic impedance at the diaphragm is given by

$$\begin{aligned} Z_h(\omega) &= \frac{A_0 v(0)}{p_f(0) + p_b(0)} \\ &= \left(\frac{-j\omega A_0}{c^2\rho_0}\right)\frac{v(0)}{bv(0) + jk\big(v_b(0) - v_f(0)\big)} \\ &= \left(\frac{-j\omega A_0}{c^2\rho_0}\right)\cos(k\ell) + \left(b + \frac{j\omega A(\ell)}{Z(\omega)c^2\rho_0}\right)\frac{\sin(k\ell)}{k}\left(\frac{\omega}{c}\right)^2\frac{\sin(k\ell)}{k} + \frac{j\omega A(\ell)}{Z(\omega)c^2\rho_0}\left(b\frac{\sin(k\ell)}{k} - \cos(k\ell)\right) \end{aligned} \tag{G17}$$

where

$$\lim_{\omega\to bc} Z_h(\omega) = \left(\frac{jbA_0}{c\rho_0}\right) \frac{1 + b\ell\left(1 + \frac{jA(\ell)}{Z(bc)c\rho_0}\right)}{1 - b\ell\left(1 + \frac{jA(\ell)}{Z(bc)c\rho_0}\right)} \quad .$$

When $\omega < bc$ one can repeat the above steps starting from

$$v_f(z) = e^{-(b+\alpha)z} v_f(0) \quad \text{and} \quad v_b(z) = e^{-(b-\alpha)z} v_b(0)$$

where

$$\alpha = \sqrt{b^2 - \omega^2/c^2} \quad . \tag{G18}$$

The resulting transmission gain is given by

$$T_h(\omega) = \frac{e^{b\ell}}{\cosh(\alpha\ell) + \left(b + \frac{j\omega A(\ell)}{Z(\omega)c^2\rho_0}\right)\frac{\sinh(\alpha\ell)}{\alpha}} \tag{G19}$$

and input impedance by

$$Z_h(\omega) = \left(\frac{-j\omega A_0}{c^2\rho_0}\right) \frac{\cosh(\alpha\ell) + \left(b + \frac{j\omega A(\ell)}{Z(\omega)c^2\rho_0}\right)\frac{\sinh(\alpha\ell)}{\alpha}}{\left(\frac{\omega}{c}\right)^2 \frac{\sinh(\alpha\ell)}{\alpha} + \frac{j\omega A(\ell)}{Z(\omega)c^2\rho_0}\left(b\frac{\sinh(\alpha\ell)}{\alpha} - \cosh(\alpha\ell)\right)} \quad . \tag{G20}$$

The following graph shows the transmission gain for a horn using $bc = 1$, a mouth diameter of $\frac{1}{2}$ wavelength and an expansion ratio of $b\ell = \ln 10$. The dashed line shows $A(\ell)/Z(\omega)$ for the mouth opening. Note that the gain at $\omega = bc$ is much less than the passband gain due to the ripple.

Other Horn Shapes

Most horns can be treated as approximations to an exponential horn. Specifically, the cross-section area can be approximated as

$$A(z) \approx A(z_0)e^{2b(z_0)(z-z_0)}$$

in the vicinity of z_0, where the exponent is given by

$$b(z) = \left(\frac{1}{2A(z)}\right)\frac{dA}{dz} \tag{G21}$$

and is assumed to vary much slower than $A(z)$. One simply has to integrate the effect of the variable $b(z)$ over the length of the horn.

For a simple cone (megaphone), we have $A(z) = A_0(1 + az)^2$ for some slope a, and the exponent is given by

$$b(z) = \frac{a}{1 + az} \quad .$$

The largest, and therefore limiting, value is $b(0) = a$.

There are many possible horn shapes which can satisfy the condition

$$b(z) < \frac{\omega}{c} \quad \forall z > 0 \quad .$$

The exponential horn is simply the shortest horn to do so.